Math 1552

Section 4.5:
L'Hopital's Rule

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Options for review session time next Wednesday:

- 4:30-6:30PM
- 5-7PM
- 6-8 PM
- 7-9 PM

→ Fill out the TP poll today by 12:45

Indeterminate Forms

$$\frac{0}{0}, \frac{\infty}{\infty}$$

can directly apply L'Hopital's rule

$$1^{\infty}, 0^{0}, \infty^{0}$$

$$0 \cdot \infty, \infty - \infty$$

looking to evaluate limits with an initial indeterminate form

L'Hopital's Rule

Let *f* and *g* be two functions. Then IF:

a) *f* and *g* are differentiable,

b) $\frac{f(x)}{g(x)}$ has the indeterminate form of $\frac{0}{0}$ OR $\frac{\infty}{\infty}$] i-portant, Cannot apply the rule without one of these forms

c) $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$, exists

THEN: $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = L$$

Evaluate the limit: $\lim_{x \to 0} \frac{3^x - 1}{4^x - 1} \frac{f}{g}$

A. 0

B. 1

C. ln(3/4)

D. $\boxed{\text{(ln3)/(ln4)}}$

$3^0 - 1 = 1 - 1 = 0$

$4^0 - 1 = 1 - 1 = 0$

$0/0 \rightarrow$ apply the rule

$3^x = e^{x \cdot \ln(3)}$

$4^x = e^{x \cdot \ln(4)}$

For any value c over the reals:

$e^{\ln(c)} = c$

$\ln(e^c) = c$

$\frac{d}{dx}\left[3^x - 1\right]$

$$= \frac{d}{dx}\left[e^{x\ln(3)} - 1\right]$$

$$= \ln(3) \cdot 3^x$$

$$\underline{\frac{d}{dx}\left[4^x - 1\right]} = \ln(4) \cdot 4^x$$

$$\text{So } \lim_{x\to 0} \frac{3^x - 1}{4^x - 1} = \lim_{x\to 0} \frac{\ln(3)\cdot 3^x}{\ln(4)\,4^x}$$

(Now can evaluate the limit directly)

$$\frac{3^0}{4^0} = \frac{1}{1}$$

$$= \frac{\ln(3)}{\ln(4)}$$

Example 2.1: Use L'Hopital's rule and logarithms to evaluate the following limit.

$$L = \lim_{x\to 0^+} x^{\underbrace{\frac{1}{\ln(5x)}}_{f(x)}} \quad 0^{-\infty}\,?$$

Logarithm rule: $\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln(f(x))} = \exp\left(\lim_{x\to a} \ln(f(x))\right)$

$$L = \exp\left(\lim_{x\to 0^+} \frac{1}{\ln(5x)} \cdot \ln(x)\right)$$

$$\ln(L) = \lim_{x\to 0^+} \frac{\ln(x) \quad f(x)}{\ln(5x) \quad g(x)} \quad \frac{-\infty}{-\infty}$$

$\to$ apply the rule

$$\exp(x) \equiv e^x$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{1}{5x} \cdot 5}$$

$$= \lim_{x \to 0^+} 1 = 1$$

$$\ln(L) = 1 \Rightarrow L = e$$

Example 2.2: Use L'Hopital's rule and logarithms to evaluate the following limit.

(know this limit) $L = \lim_{x\to\infty} \boxed{\left(1+\frac{a}{x}\right)^x}$, a is any constant 1^∞

$f(x)$

Logarithm rule: $\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln(f(x))} = \exp\left(\lim_{x\to a} \ln(f(x))\right)$

$$L = \exp\left(\lim_{x\to\infty} x\cdot\ln\left(1+\frac{a}{x}\right)\right)$$

$$\Longleftrightarrow \ln(L) = \lim_{x\to\infty} x\cdot\ln\left(1+\frac{a}{x}\right)$$

$$\to \infty\cdot\ln(1) \to \infty\cdot 0$$

→ how to apply L'Hopital's rule?

$$\ln(L) = \lim_{x \to \infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} \qquad \frac{0}{0} \checkmark$$

$$= \lim_{x \to \infty} \frac{\frac{1}{1+\frac{a}{x}} \cdot \left(-\frac{a}{x^2}\right)}{-\frac{1}{x^2}}$$

$$\frac{d}{dx}\left[\ln\left(1+\frac{a}{x}\right)\right] = \frac{1}{1+\frac{a}{x}} \cdot \frac{d}{dx}\left[1+\frac{a}{x}\right]$$

$$= \frac{1}{1+\frac{a}{x}} \cdot \left(\frac{-a}{x^2}\right)$$

$$= \lim_{x \to \infty} \frac{a}{1 + \underbrace{\frac{a}{x}}_{\to 0}} = a$$

$$\to \ln(L) = a \Rightarrow L = e^{a}$$

What about this limit?

$$L_1 = \lim_{x \to \infty} \left(1 + \frac{a}{x^2}\right)^{x^2} \longrightarrow e^a$$

$$\left(1 + \frac{2}{x} + \frac{4}{x^2}\right)^{x^2}$$

(work out the details on your own)

$$L_2 = \lim_{N \to 0} \left(1 + bN\right)^{\frac{1}{N}} \longrightarrow e^b$$

($b = 2$ on the next slide)

Evaluate the limit: $L = \lim_{x\to 0^+} (1+2x)^{\frac{1}{x}}$ 1^∞

A. $\boxed{e^2}$

B. $e^{1/2}$

C. 1

D. Infinity

$$\ln(L) = \lim_{x\to 0^+} \frac{\ln(1+2x)}{x} \quad \frac{0}{0} \checkmark$$

$$= \lim_{x\to 0^+} \frac{\frac{2}{1+2x}}{1}$$

$$= \lim_{x\to 0^+} \frac{2}{1+2x} = 2$$

So $\ln(L) = 2 \longleftrightarrow L = e^2$

Compendia of Common Limits (memorize)

1) If $x > 0$, then $\lim_{n\to\infty} x^{1/n} = 1$.

2) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$. ex: $\lim_{N\to\infty} \left(\frac{1}{2}\right)^N = 0$ $\left(x = \frac{1}{2}\right)$

3) If $\alpha > 0$, then $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$. ex: $\lim_{N\to\infty} \frac{1}{N^2} = 0$ $(\alpha = 2)$

4) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

5) $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

6) $\underbrace{\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x}_{\text{from The Previous example}}$

7) $\lim_{n\to\infty} n^{1/n} = 1$

What is the value of the following limit:

$$L = \lim_{N\to\infty} (2N)^{1/N}$$

$$= \lim_{N\to\infty} \underbrace{2^{\frac{1}{N}}}_{1 \text{ by } (1)} \cdot \underbrace{N^{1/N}}_{1 \text{ by } (7)} = 1$$

Extra Problem I: Evaluate the following limit:

$$L = \lim_{w \to -6} \frac{\sin(2\pi w)}{w^2 - 36}$$

→ work this problem out on your own

→ solution: $L = \frac{-\pi}{6}$

Extra Problem II: Evaluate the following limit:

$$L = \lim_{x \to 0^+} \frac{\sin(2x)}{2x} \quad , \quad \lim_{x \to 0^+} \frac{\sin(ax)}{ax} = 1$$

→ work this problem on your own

→ solution: $L = 1$

→ compare to the familiar limit: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

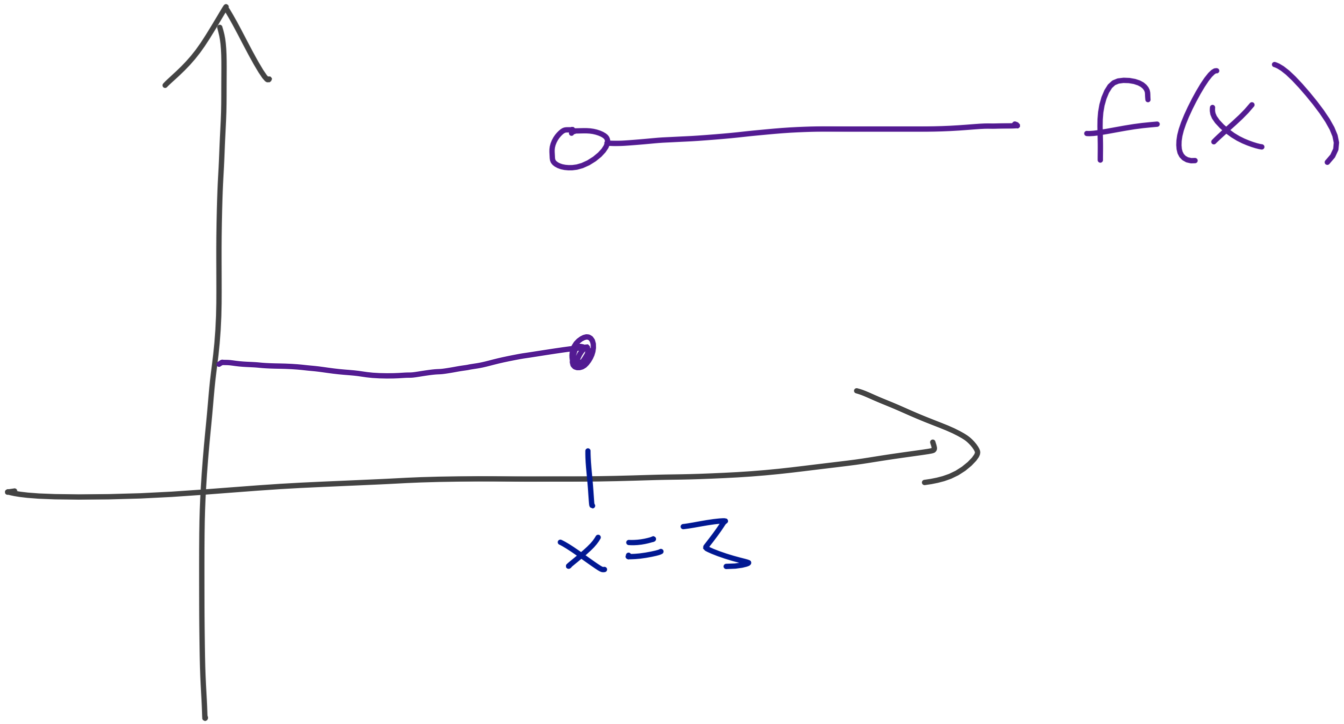
Extra Problem III: Evaluate the following limit:

$$L = \lim_{x \to \frac{1}{2}^+} \left(x - \frac{1}{2}\right) \tan(\pi x)$$

→ work this out on your own

→ solution: $L = -\frac{1}{\pi}$

examples of when limits do not exist:



$$\lim_{x \to 3} f(x) \text{ DNE}$$

Bonus Practice Problems: Evaluate each of the following limits:

(In class: practice verifying that we get an indeterminate form in each case)

- $$\lim_{x\to\infty} \frac{x^2-2}{2x^2-3x+1}$$ $\frac{\infty}{\infty}$

  Hint: Multiply through by $1 = \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$, and then take limits

- $$\lim_{t\to+\infty} \left[t \cdot \ln\left(1+\frac{8}{t}\right)\right]$$ $\infty \cdot 0 \rightarrow$ transform to apply the rule:

  $$\lim_{t\to+\infty} \frac{\ln\left(1+\frac{8}{t}\right)}{1/t} \quad \frac{0}{0}$$

3. $$\lim_{x\to 0^+} \frac{3^x-4^x}{x^2-2x}$$

4. $$\lim_{x\to+\infty} \left[\sqrt{x^2+2}-\sqrt{x+2}\right]$$

   Hint: Multiply through by the *conjugate* $1 = \frac{\sqrt{x^2+2}+\sqrt{x+2}}{\sqrt{x^2+2}+\sqrt{x+2}}$, to simplify the numerator first

5. $$\lim_{x\to 0} x^{3x}$$

$$\lim_{x\to 0^+} \frac{3^x - 4^x}{x^2 - 2x}$$

$\frac{0}{0}$ → apply the rule directly

$$\lim_{x\to+\infty} \left[\sqrt{x^2+2} - \sqrt{x+2}\right]$$

$\infty - \infty$

Simplify by writing:

$$\left(\sqrt{x^2+2} - \sqrt{x+2}\right) \cdot \frac{\left(\sqrt{x^2+2} + \sqrt{x+2}\right)}{\left(\sqrt{x^2+2} + \sqrt{x+2}\right)}$$

$$= \frac{(x^2+2-(x+2))}{\sqrt{x^2+2}+\sqrt{x+2}} = \frac{(x^2-x)}{\sqrt{x^2+2}+\sqrt{x+2}}$$

So as $x \to +\infty$: $\frac{\infty}{\infty}$

$0 \cdot (-\infty)$ X

$L = \lim_{x\to 0^+} x^{3x}$ $\quad 0^0$, $\ln(L) = \lim_{x\to 0^+} 3x \cdot \ln(x)$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{3x}} \quad \frac{-\infty}{\infty}$$

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Section 8.8:
Improper Integrals

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Today’s Learning Goals

- Be able to identify when an integral is improper (three cases)
- Rewrite an improper integral as a limit
- Understand the meaning of convergence and divergence as relating to integration
- Evaluate improper integrals

Improper integrals

$$\int_a^b f(x)\,dx$$

A definite integral is improper if:

$$\int_0^{2\pi} \tan(x)\,dx$$

$\to \tan(\pi/2) = +\infty$

$\to \tan(3\pi/2) = -\infty$

- The function has a vertical asymptote at $x=a$, $x=b$, or at some point c in the interval (a,b).
- One or both of the limits of integration are infinite (positive or negative infinity).

$$\int_{-\infty}^{\infty} f(x)\,dx, \quad \int_1^{\infty} g(x)\,dx$$

Which of the following integral(s) is (are) improper? Why / which case?

$$\checkmark\ 1)\int_0^{\frac{\pi}{4}} \tan(2x)dx$$

$$\tan\left(2 \cdot \frac{\pi}{4}\right) = +\infty$$

$$\lim_{x \to \frac{\pi}{4}} \tan(2x) = +\infty$$

$$2)\int_{-1}^{1} \frac{x-3}{x^2-2x-3}dx$$

$$3)\int_0^{\frac{\pi}{2}} \cos(x)dx$$

$$4)\int_0^{3} \frac{x-2}{x^2-6x+8}dx$$